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# Computational Issues Associated With Gear Rattle Analysis Part II : Evaluation Criteria for Numerical Algorithms

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## Summary

The main focus of this study is to establish evaluation criteria for direct time domain integration algorithms used to solve gear rattle type problems. Such criteria may be used to identify specific numerical problems encountered. The ultimate goal obviously is to find reasonably accurate and reliable methods of solution for such physical systems. Six case studies of increasing complexity, linear to highly non-linear, are solved using well known algorithms. The solutions to the linear model are verified by using analytical results. Non-linear model solutions as yielded by different algorithms are compared qualitatively and quantitatively. Several non-linear simulation models have been validated by comparing predictions with experimental data and results available in the literature.

## Nomenclature

| | |
|---|---|
| b | breakpoints of nonlinearity |
| C | damping coefficient |
| e | static transmission error |
| f | nonlinear function for stiffness |
| g | nonlinear function for damping |
| $\hat{h}_i$ | initial average trial step size |
| $h^*$ | Bulirsch-Stoer method substep size |
| J | Jacobian |
| K | stiffness element |
| R | radius |
| $R_\lambda$ | eigenvalue ratio (largest to lowest non-zero) |
| t | time |
| T | torque |
| x | length |
| y | dependent variable vector |
| $\alpha$ | strength of stiffness nonlinearity |
| $\beta$ | strength of damping nonlinearity |
| $\delta$ | relative displacement |
| $\varepsilon$ | amplitude of transmission error |
| $\eta$ | elements of non-dimensional damping matrix |
| $\kappa_{r,s}$ | solution coherence between r and s |
| $\Omega$ | elements of non-dimensional stiffness matrix |
| $\Omega_e$ | non-dimensional input frequency |
| $\omega$ | non-dimensionalizing frequency |
| $\lambda$ | eigenvalue |
| $\Gamma_r$ | contact ratio |
| $\psi$ | angular displacement |
| $\theta$ | absolute displacement |
| $\Xi$ | normal mode matrix |
| $\varphi$ | normal co-ordinates |
| $\zeta$ | modal damping ratio |

*Subscripts*

| | |
|---|---|
| a | alternating component |
| c | characteristic quantity |
| d | drag component |
| e | excitation component |
| i | element number |
| j | harmonic number |
| m | mean component |

*Superscripts*

| | |
|---|---|
| $(\bar{\ })$ | dimensional quantity |
| ( )' | non-dimensional velocity |
| ( )" | non-dimensional acceleration |
| ( )* | complex conjugate |

*Algorithms*

| | |
|---|---|
| ADAM1 | Adam's method (IMSL, 1986) |
| ADAM2 | Adam's method (LSODE; Hindmarsh, 1974) |
| BS | Bulirsch-Stoer method |
| GEAR1 | Gear method (IMSL, 1986) |
| GEAR2 | Gear method (LSODE; Hindmarsh, 1974) |
| RKC2 | second order Runge-Kutta method |
| RKC | Classical (4th order) Runge-Kutta method |
| RKF | Runge-Kutta-Fehlberg method |
| RKV | Runge-Kutta-Verner method |

## 1. Introduction

Several numerical algorithm comparison studies have been conducted on stiff systems in order to determine which method(s) will provide reasonably accurate solutions at minimal computational cost(s) (Greenspan, 1966; Willoughby, 1974; Jackson and Sacks-Davis, 1980; Gupta, 1980; Aiken, 1985; Bert and Stricklin, 1988). Common test problems have been confined to kinetics rate equation set (Aiken, 1985), mass action kinetics, or diffusion-convection problems (Willoughby, 1974; Aiken 1985; Hairer et al., 1987). Other examples include a two-degree-of-freedom mechanical system with cubic non-linearities (D'Souza and Garg, 1984). However, a multi-degree-of-freedom vibratory system with multiple clearances, used to model gear rattle type problems, is yet to be examined critically on the same basis.

An examination of the classical definition of stiff problems has led many researchers such as Willoughby (1974) and Aiken (1985), to identify additional fundamental problems which contain complicating features that may severely restrict numerical computations. The gear rattle problem as mentioned in the companion paper (Part I) belongs to such a category.

## 2. Scope and Objectives

The purpose of this study is to numerically obtain solutions for a set of ordinary differential equations with multiple clearances; for example refer to equations (16-19 ) developed in Part I. Most numerical algorithms reduce such *N* ordinary differential equations (ODE's) into *2N* first-order ODE's of the initial value problem type as shown below.

$$dy/dt = y' = p(y, t);\ \text{given } y(a) \text{ where } a \le t \le b \qquad (1)$$

Here, t is the independent variable and y is the dependent variable vector of dimension *2N*. The nature of the ODE's governing the gear rattle problem, as seen in Part I, is highly non-linear and therefore the local eigenvalues of the system, which vary over time, may make the system appear very stiff at certain points during integration. The determination of how stiff the system may become, and how well the numerical methods cope with it is the focus of this study. Using selected operating points $y_0$, a reasonable range of $R_\lambda$ will be estimated. In addition, several versions of the gear rattle system formulated earlier in Part I, each of differing complexity, will be used to compare various well known algorithms. The steady-state responses of the example cases will be analyzed and compared with known solutions (if any are available) in an attempt to verify or justify solutions. In addition computation time, step size *h* of the independent variable, t and other algorithm specific parameters will be examined. Given the complexity of the computational problem, it is expected that the present study will be preliminary in nature. Accordingly, the scope is limited to a few example cases and only seven different algorithms.

## 3. Review of Direct Integration Methods

Before discussing the numerical methods evaluated, it is necessary to present a brief review of terms used. First note that a set of N second-order ODE's is reduced to a set of 2N first-order ODE's that can be solved by many numerical packages. Second, the derivatives of an arbitrary solution y will be denoted by $y' = p(y, t)$, and called simply function **p**. Third, the numerical techniques discussed in this study will develop a mesh of t, $a \le t \le b$ with *h* denoting the mesh step size. Lastly, the $n^{th}$ estimate of y in the mesh of t is denoted by $y_n$.

In a direct time integration method, governing equations are solved successively using a step-by-step numerical integration procedure (Gear, 1971; Shampine and Gordon, 1975). Derivatives are generally approximated using difference formulas involving one or more increments of the independent variable, t. Two basic approaches are used in direct time integration: (i) explicit and (ii) implicit methods. In an explicit formulation, the response quantity, say $y_{n+1}$, will be expressed in terms of previously determined values $y_n, y_{n-1}, \ldots, y_{n-k+1}$. It is often called a k-step method; if k=1 then it is a one-step method, and if k>1 then it is a multi-step method. In the implicit formulation, $y_{n+1}$ will in addition be a function of itself, hence this process is iterative in nature. Methods using this type of formulation may use difference equations and the solutions are calculated by solving the algebraic equations. Others use a backward differentiation formula to arrive at the system of algebraic equations (Gear, 1971; Barton, 1980; Gear, 1980). The algorithms considered for this study are the Runge-Kutta fixed step second and fourth order (RKC2, RKC), variable step size (RKF, RKV), Adams variable step and order (ADAM1, ADAM2), Gear's stiff (GEAR1 and GEAR2), and the Bulirsch-Stoer (BS) methods.

Errors enter numerical solution of initial value ODE's through two main sources: (i) truncation error, and (ii) round-off error. Truncation error is dependent on the accuracy to which a chosen method approximates the integration. Further, this type of error can be broken into local (the error of a single step), and global truncation error (the accumulation of local truncation error); see Gerald and Wheatley (1989) for details. Round-off error is due to the machine precision. The stability of an algorithm refers to the continuous dependence of a solution on the data produced by the method, as well as the propagation of the errors. A numerically stable method is one in which the round-off and truncation errors are not grossly magnified. Error estimators have been found to be valid only for small step sizes (Shampine and Gordon, 1975). Hence in some algorithms error estimators may not detect large errors which may lead to an unstable solution. Generally, instability becomes a consideration only when the accuracy requirement does not restrict step size sufficiently. If a solution changes fairly rapidly and the error criterion forces the algorithm to follow the solution, it appears that the accuracy requirement dominates invariably (Nakamura, 1991).

## 4. Example Cases

An analysis of numerical methods will be conducted using parameters from a generic automotive manual transmission (See Figure 4, Part I). Baseline inertial and stiffness parameters are incorporated into the dimensionless format of Part I (refer to equations 16-19) and these are listed in Table 1. Typical values used to define the non-linearities will be given later. Six example cases of differing complexity, from 1A-linear to 3B-extremely non-linear, are listed in Table 2. The extent of the stiffness non-linearity for each element is categorized as linear, moderately non-linear (i.e. $\alpha$>0.01) and strongly non-linear ($\alpha$<0.01). Since damping element non-linearities are associated with the stiffness changes, clearance non-linearities increase or decrease modal damping. In Table 2 a fully linear system (Case 1A) has been included so that numerical solutions can be validated by comparing with analytical results.

## 5. Evaluation Criteria

The numerical indices which may be used to evaluate an algorithm are: (i) computational performance indices for a given algorithm, and (ii) indices used to compare the solutions provided by different algorithms for the same physical problem. In the first category the commonly used criteria are the following:

(a). *Total time required for the solution* : This may represent a useful comparison from a users viewpoint since an excessive cpu time may slow down research.

(b). *Number of steps and function calls* : These two indices are closely tied to the total time. However, an unusually high number of steps or function calls may be an indicator of an algorithm having difficulty obtaining solutions near critical points, say at the physical discontinuity of the system model.

(c). *Tolerance* : This is a user defined parameter to ensure a certain accuracy of solution. Each algorithm attempts to control the norm of the local error so that the error is proportional to the tolerance. Obviously, a solution with a high tolerance value requires a smaller $h$ and longer cpu times than that with a low tolerance.

For the second category four main criteria have been developed to compare the solutions provided by the different algorithms for the gear rattle type problems:

(a). *Time domain results* : Time history output is conveniently placed in the form of a graph with abcissa-normalized time to reflect cycles of fundamental period corresponding to the excitation frequency. Thus super or sub harmonic responses are easily ascertained.

(b). *Transmissibility* TR: This is a numerical measure of the agreement between time domain solutions and is defined as follows:

$$TR_{rms} = \frac{\theta''_{4_{rms}}}{\theta''_{1_{rms}}}; \quad \theta_{i_{rms}} = \left[\frac{1}{T}\int_0^T (\theta''_i)^2\, dt\right]^{0.5}, i = 1,4 \qquad (2, 3)$$

$$TR_{peak} = \frac{\theta''_{4_{peak}}}{\theta''_{1_{peak}}} \qquad (4)$$

where the subscript peak refers to the peak to peak value of the acceleration over the period of interest. This index is applied only to the steady state time domain solutions with rms and peak TR values computed from a span of eight cycles.

(c). *Spectral Contents* : The frequency spectrum enables quick identification of primary response frequencies.

(d). *Solution coherence* $\kappa_{xy}(f)$: Since vibro-impacts induce broadband frequency response, one finds difficulty in the comparison of frequency domain information. Hence, in order to determine the agreement of the solutions , only selected frequencies or bandwidths were used in the determination of $\kappa_{xy}(f)$, which is defined as $|(P_{xy}(f))^2|/P_{xx}(f)P_{yy}(f)$ where $P_{xx}(f)$, $P_{yy}(f)$ and $P_{xy}(f)$ are the averaged auto, auto and cross power spectral densities respectively. Typical signal processing includes 128 points per time window and four ensembles.

## 6. Results

Six case studies as identified in Table 2 are presented, covering a range of complexity regarding the number of non-linear elements and types of loading; also refer to Figure 4 of Part I. The typical excitation in each case consists of a mean load and five harmonics of sinusoidal load applied to the inertial element $I_1$, a mean load applied to $I_4$, and light drag loads applied to all elements. Additionally, cases 2B and 3B include transmission error excitation applied at the mesh frequency $20\Omega_{e11}$ where $\Omega_{e11}$ is the frequency of the main shaft rotation.

Tables 3 through 5 and Figures 1 to 3 show performance criteria and typical time and frequency domain responses for a physical system of increasing complexity from linear (case 1A) to a system with three clearance type non-linearities with multiple excitations (case 3B). Note from the tables that the computational time increases as the number of non-linearities increase, as expected. The increased times of RKC2 and RKC are half that of the variable step routines, while still maintaining compatible solutions. This is because the variable step method hunts for an optimal $h$, thus leading to extra time. Since the RKC2 method is similar to the Euler method it performs better in dealing with eigenvalues which are highly oscillatory. Higher order methods may become locally unstable for similar eigenvalues, necessitating stepsize adjustments (Gaffney, 1984) as evident from Table 6. Here, the solution coherences for case 2B show that the fixed step algorithms matched very well with more time consuming methods. The variable stepping methods generally show a larger initial average trial step size $\hat{h}_i$, but it is noted that this trial size nearly always fails near the breakpoints of the physical non-linear functions (See Figure 5, Part I). The step size reductions that ensue require much code and repetition, thus increasing the computing time. Note that for case 3B when e(t) is the dominant excitation, the Bulirsch-Stoer (BS) method fails to converge. This was the only routine, of all seven tested, that consistently displayed convergence failures and incompatible solutions, especially for systems with multiple non-linearities.

Numerical indices given in Table 6 indicate two trends. First, the coherence degrades for the solutions far away from the driving point of the torque or transmission error excitation. This phenomena can be seen in Figure 4. Note that the solution about $I_2$ appears to be more difficult to acquire than at other inertial points. This could be attributed to the relationship between stiffnesses and mass loading effect upon $I_2$, combined with the stabilizing effect of the mean load on $I_4$. The second trend seen between algorithms is that the TR yielded by the fixed step routines appears to encompass a larger bandwidth (rms to peak) than those by higher order algorithms. This may be due to the inherent smoothening effect higher order methods impose.

Loading effects are also examined. For case 3A, the ratio of the mean to alternating load is set to 0.05 for the lightly loaded case and 0.5 for the moderately loaded case. An increase in the mean load on the system reduces the computing time for variable step algorithms by about 30%. Typical time and frequency domain plots are shown in Figure 5. The solution coherences are also improved for such moderately loaded systems.

## 7. Validation and Conclusion

This evaluation of numerical integration methods for clearance-type non-linear geared torsional systems is believed to be the first of its kind. However it is a limited study and therefore any conclusions derived here must be considered as preliminary and qualitative since only seven different algorithms have been evaluated.

In order to validate the proposed numerical techniques for non-linear systems, results were compared with available experimental data from a four-degree-of-freedom system consisting of one gear pair. Comparison of analytic vs. experimental angular acceleration of the loaded input inertia and output gear are found to be as follows given in terms of % deviation from the experimental data:

| gear motion | input inertia | output gear |
|---|---|---|
| single-sided impacts | 2.7% | 3.1% |
| double-sided impacts | 1.4% | 7.8% |

These results are very promising and indicate that the techniques presented in this study are indeed suitable.

The main conclusion of this study is that the choice of an algorithm is dictated by the type of physical system, including non-linearities, under evaluation. For instance, the higher order methods appear to become computationally inefficient as evident by excessive cpu time and functional calls when the system becomes one or any combination of the following: lightly loaded, lightly damped, highly non-linear, or strongly stiff. Such computational inefficiency does not necessarily mean that the solutions obtained are inconsistent with other data, only that the lower order fixed step methods operate faster the higher order variable steppers. Numerical stiffness becomes an issue for all algorithms tested since a system with clearance type non-linearities has the potential to become strongly stiff, forcing numerical integration schemes of the time marching type to severely restrict step size. A variable step or variable order algorithm typically senses the physical discontinuity and in the attempt to step across it, "bottoms out", reducing h to the prescribed user tolerance. Conversely the fixed step methods tend to walk through the discontinuities and have the ability to converge quickly after the step, in most cases (see case 3B). This appears to be one of the main reasons why fixed stepping routines ran faster in nearly every case. This feature has been noted in the literature (Chi and Tucker, 1982; Chi, 1984; Cash and Karp, 1990), and may be due to lesser inherent damping of the low order algorithms. Thus, new algorithms incorporating order reduction integration models, when discontinuities are evident, have begun to appear (Chi and Tucker, 1982; Chi, 1984; Cash and Karp, 1990; Prestl, 1991).

## 8. Recommendation and Future Research

Guidelines for choosing an appropriate algorithm to solve the gear rattle type problems is as follows, based on this limited study.

a. Diagnose the nature of the non-linear system; see Part I for details.
b. Classify the stiffness of the corresponding linear system by using the contact ratio concept as given in Part I.
c. Determine which solution domain is of primary importance.
d. Use evaluation criteria developed here to ensure that an appropriate algorithm is being used.

A new or unknown physical system under study must be evaluated using at least two different algorithms with several tolerance specifications. If solutions from different algorithms do not seem to converge to a common set, check for numerical creep or other numerical problems.

This numerical study is preliminary in nature since only seven well known algorithms were tested. However, it has been shown that evaluation of clearance-type non-linear torsional systems relies critically upon choice of algorithm, and some of the pitfalls one may encounter while investigating such a system have been documented. Future work should be directed to evaluating implicit, variable step, and variable order schemes with the capability of detecting a physical discontinuity and stepping through it by switching the order of the algorithm.

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**Table 1 ***

**Dimensionless Parameters for Numerical Study**

| $\xi_{11}$ | $\xi_{22}$ | $\xi_{33}$ | $\xi_{44}$ |
|---|---|---|---|
| $3.333 \times 10^{-6}$ | 0.002 | $9.524 \times 10^{-5}$ | $1.667 \times 10^{-4}$ |

| $\eta_{11}$ | $\eta_{12}$ | $\eta_{22}$ |
|---|---|---|
| 0.005 | 3.333 | 0.111 |
| $\eta_{23}$ | $\eta_{33}$ | $\eta_{34}$ |
| 0.005 | 7.143 | 50.000 |
| $\Omega_{11}^2$ | $\Omega_{12}^2$ | $\Omega_{22}^2$ |
| 0.222 | 148.148 | $1.482 \times 10^3$ |
| $\Omega_{23}^2$ | $\Omega_{33}^2$ | $\Omega_{34}^2$ |
| 63.492 | $7.619 \times 10^3$ | $5.333 \times 10^4$ |

**Table 2 ***

**Case Study Models**

| Case | Element features | index i | $\alpha_i$ | $\beta_i$ | $b_i$ | $\varepsilon$ | Comments |
|---|---|---|---|---|---|---|---|
| 1A | linear | 1 | 1.0 | 1.0 | ∞ | 0.0 | fully linear model |
| | linear | 2 | 1.0 | 1.0 | ∞ | 0.0 | |
| | linear | 3 | 1.0 | 1.0 | ∞ | 0.0 | |
| 2A | m NL | 1 | 0.2 | 0.25 | 1.0 | 0.0 | clutch moderately |
| | linear | 2 | 1.0 | 1.0 | ∞ | 0.0 | non-linear |
| | linear | 3 | 1.0 | 1.0 | ∞ | 0.0 | |
| 3A | linear | 1 | 1.0 | 1.0 | ∞ | 0.0 | highly non-linear |
| | linear | 2 | 1.0 | 1.0 | ∞ | 0.0 | gear backlash, spectral |
| | backlash | 3 | 0.0 | 0.0 | 1.0 | 0.0 | coupling possible |
| 1B | m NL | 1 | 0.1 | 0.25 | 160.0 | 0.0 | two non-linear |
| | linear | 2 | 1.0 | 1.0 | ∞ | 0.0 | oscillators connected |
| | backlash | 3 | 0.0 | 0.0 | 1.0 | 0.0 | by a linear oscillator |
| 2B | m NL | 1 | 0.01 | 0.025 | 160.0 | 0.0 | three non-linear |
| | backlash | 2 | 0.0 | 0.0 | 58.67 | 0.0 | oscillators, includes |
| | backlash | 3 | 0.0 | 0.0 | 1.0 | 0.1 | transmission error |
| 3B | s NL | 1 | 0.001 | 0.01 | 160.0 | 0.0 | three non-linear |
| | backlash | 2 | 0.0 | 0.0 | 58.67 | 0.0 | oscillators, severe |
| | backlash | 3 | 0.0 | 0.0 | 1.0 | 0.1 | coupling possible |

m NL: moderately non-linear
s NL: strongly non-linear

* Refer to Figures 4 and 5 of Part I and the Nomenclature for the identification of variables.

**Table 3**

**Algorithm Performance Characteristics**

**Case 1A, Reference: Analytical Modal Expansion**

| Algorithm | Average step $\hat{h}_i$ | Number of steps | Function calls | Run time hour:min:sec | $\kappa_{\theta_i^*,\theta_{i+4}^*}$ (all) | TR (rms:peak) |
|---|---|---|---|---|---|---|
| RKC2 | 0.0026 | 192000 | 384000 | 00:05:57 | 1.0 | 1.0 : 1.0 |
| RKC | 0.0256 | 19200 | 76800 | 00:01:03 | 1.0 | 1.0 : 1.0 |
| RKF | 0.0440 | 7116 | 54022 | 00:01:40 | 1.0 | 1.0 : 1.0 |
| RKV | 0.0413 | 9488 | 78697 | 00:02:00 | 1.0 | 1.0 : 1.0 |
| GEAR1 | 0.0188 | 45960 | 70240 | 00:03:24 | 1.0 | 1.0 : 1.0 |
| GEAR2 | 0.0188 | 45920 | 70110 | 00:03:11 | 1.0 | 1.0 : 1.0 |
| ADAM1 | 0.0161 | 38140 | 70770 | 00:03:00 | 1.0 | 1.0 : 1.0 |
| ADAM2 | 0.0162 | 38130 | 70750 | 00:02:58 | 1.0 | 1.0 : 1.0 |
| BS | 0.0158 | 31022 | 676322 | 00:13:52 | 1.0 | 1.0 : 1.0 |

**Table 4**

**Algorithm Performance Characteristics**

**Case 2A**

| Algorithm | Average step $\hat{h}_i$ | Number of steps | Number of function calls | Run time hour:min:sec |
|---|---|---|---|---|
| RKC2 | 0.0026 | 256000 | 512000 | 00:09:56 |
| RKC | 0.0256 | 25600 | 102400 | 00:03:24 |
| RKF | 0.0132 | 44251 | 486761 | 00:09:34 |
| RKV | 0.0157 | 37085 | 360268 | 00:07:19 |
| GEAR1 | 0.0080 | 92630 | 158900 | 00:08:32 |
| ADAM1 | 0.0081 | 93340 | 152500 | 00:07:48 |
| BS | 0.0159 | 41116 | 891140 | 00:17:26 |

**Table 5**
**Algorithm Performance Characteristics**
**Case 3B, ε=0.10**

| Algorithm | Average step $\hat{h}_i$ | Number of steps | Number of function calls | Run time hour:min:sec |
|---|---|---|---|---|
| RKC2 | 0.001 | 800000 | 1600000 | 00:25:47 |
| RKC | 0.0128 | 64000 | 256000 | 00:17:02 |
| RKF | 0.0400 | 125034 | 1150312 | 00:23:24 |
| RKV | 0.0377 | 170441 | 2090359 | 00:50:03 |
| GEAR1 | 0.0404 | 246600 | 921000 | 00:19:55 |
| ADAM1 | 0.0365 | 411500 | 678400 | 00:33:59 |
| BS | ****** | ****** | ****** | ****** |

****** algorithm failed to converge

**Table 6**
**Solution Coherence of Case 2A**
**Reference: RKC2**

| Algorithm | | $K_{\theta_1^*,\theta_{1,d}^*}$ | $K_{\theta_2^*,\theta_{2,d}^*}$ | $K_{\theta_3^*,\theta_{3,d}^*}$ | $K_{\theta_4^*,\theta_{4,d}^*}$ | TR (rms:peak) |
|---|---|---|---|---|---|---|
| RKC2 | [1-5] [1-10] | reference | reference | reference | reference | 0.32 : 0.32 |
| RKC | [1-5] | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.32 : 0.32 |
| | [1-10] | 1.0000 | 0.9947 | 0.9999 | 0.9999 | |
| RKF | [1-5] | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.32 : 0.32 |
| | [1-10] | 1.0000 | 0.9998 | 1.0000 | 1.0000 | |
| RKV | [1-5] | 1.0000 | 0.9997 | 1.0000 | 1.0000 | 0.31 : 0.34 |
| | [1-10] | 1.0000 | 0.9769 | 0.9997 | 0.9996 | |
| GEAR1 | [1-5] | 1.0000 | 0.9993 | 1.0000 | 1.0000 | 0.31 : 0.34 |
| | [1-10] | 1.0000 | 0.9492 | 0.9997 | 0.9995 | |
| ADAM1 | [1-5] | 1.0000 | 0.9993 | 1.0000 | 1.0000 | 0.31 : 0.34 |
| | [1-10] | 1.0000 | 0.9647 | 0.9996 | 0.9991 | |
| BS | [1-5] | 1.0000 | 0.9994 | 1.0000 | 1.0000 | 0.31 : 0.34 |
| | [1-10] | 1.0000 | 0.9671 | 0.9997 | 0.9996 | |

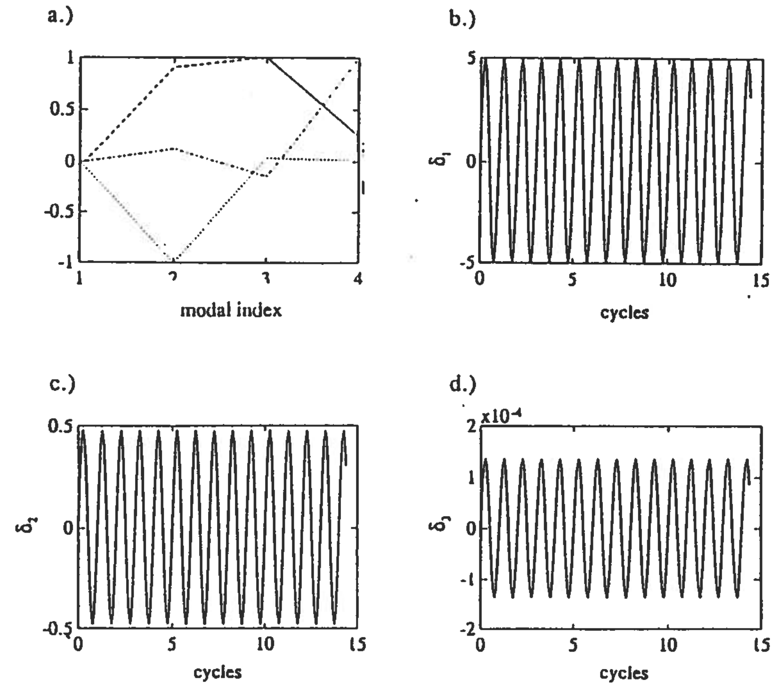


Figure 1. a.) Normal modes of the system, ——, mode 1, ---, mode 2, ···, mode 3, -·-, mode 4. b-d.) Time histories of normal mode expansions.

See the Nomenclature for the identification of the variables.

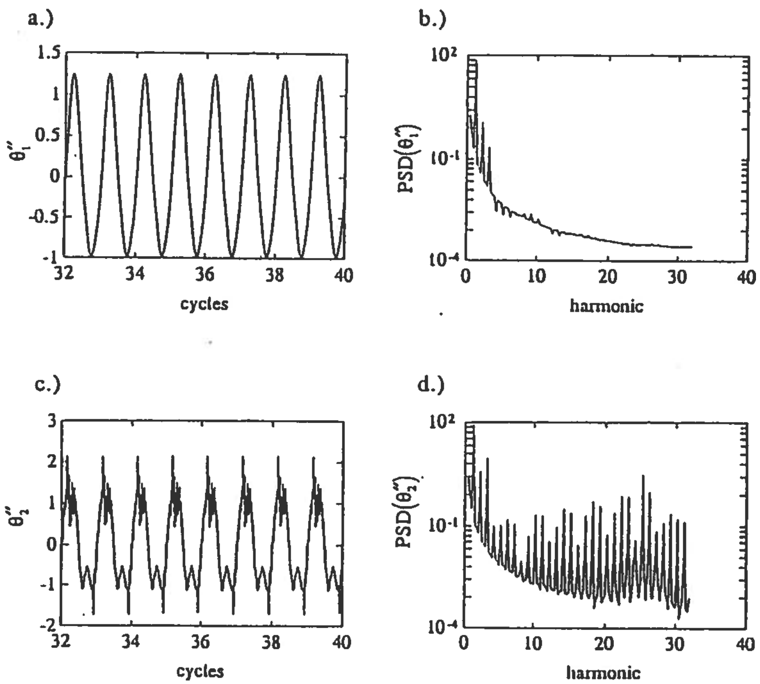


Figure 2 a-d.) Time histories and power spectral densities of the first two absolute accelerations for Case 2A.

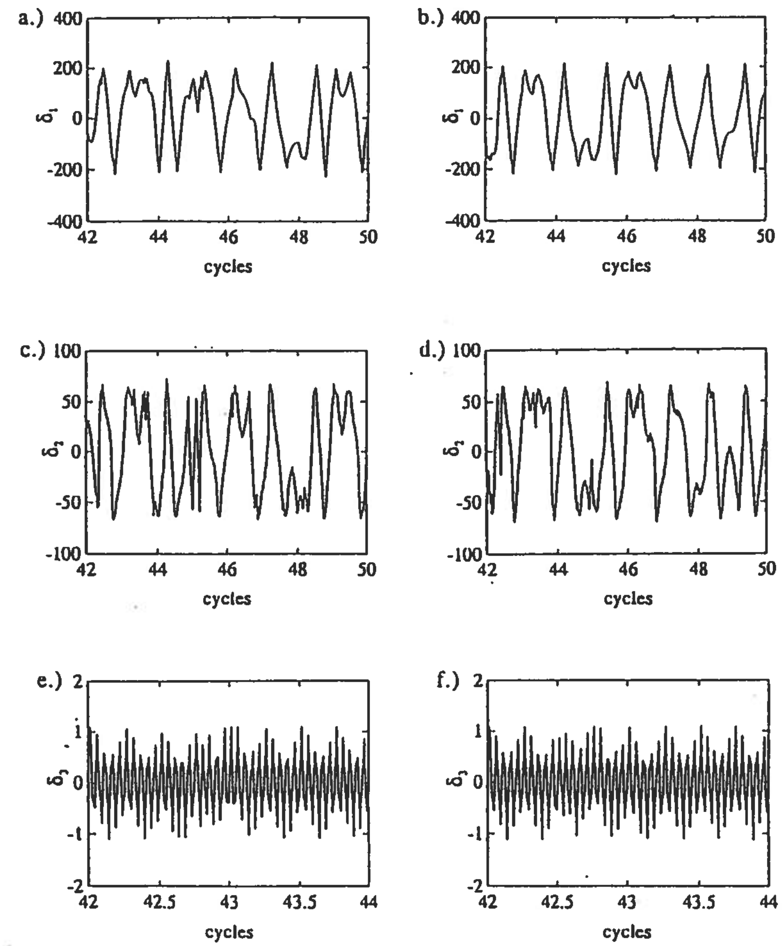


Figure 3. Time histories of the relative displacements for Case 3B using:
a, c, e.) Second-order Runge-Kutta algorithm.
b, d, f.) Runge-Kutta-Verner algorithm.

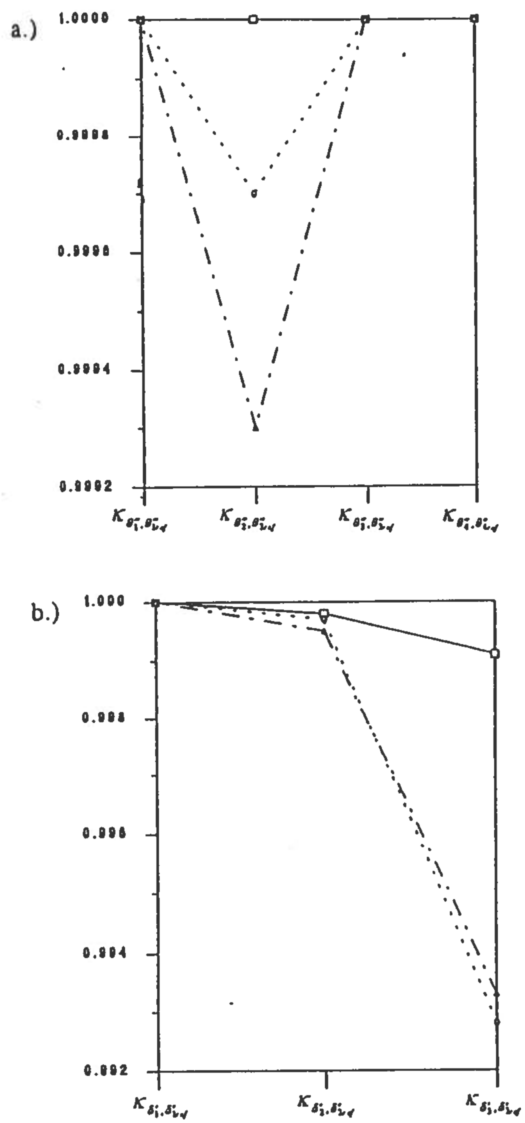

Figure 4. The solution coherence factors for Case 2A, averaged over first 5 harmonics of $\Omega_{e_{11}}$, of a.) absolute accelerations, b.) relative velocities.

—□—, Fourth-order Runge-Kutta, —◇—, Runge-Kutta-Verner, —△—, Gear.

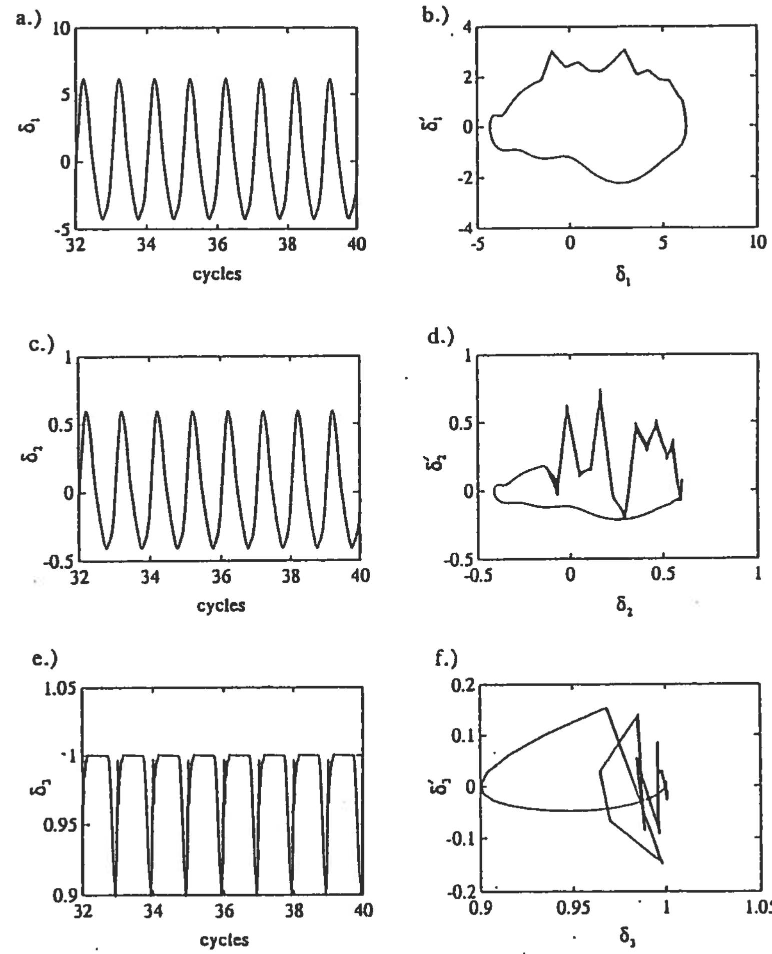


Figure 5 a-f.) Time histories and phase planes for Case 3A, moderately loaded.